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VOLUMES OF GEODESIC BALLS IN HEISENBERG GROUPS $\mathbb{H}^5$

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Abstract. Let $\mathbb{H}^5$ be the 5-dimensional Heisenberg group equipped with a left-invariant metric. In this paper we calculate the volumes of geodesic balls in $\mathbb{H}^5$. Let $B_e(R)$ be the geodesic ball with center e (the identity of $\mathbb{H}^5$) and radius R in $\mathbb{H}^5$. Then, the volume of $B_e(R)$ is given by

$$\begin{aligned} &Vol(B_e(R)) \\ =&\frac{4\pi^2}{6!}\Bigg(p_1(R)+p_4(R)\sin R+p_5(R)\cos R+p_6(R)\int_0^R\frac{\sin t}{t}dt \\ &\qquad +q_4(R)\sin(2R)+q_5(R)\cos(2R)+q_6(R)\int_0^{2R}\frac{\sin t}{t}dt\Bigg) \end{aligned}$$

where p_n and q_n are polynomials with degree n.

1. Introduction

Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $<,>$ and N its unique simply connected 2-step nilpotent Lie group with the left invariant metric induced by $<,>$ on $\mathcal{N}$. Let $\mathcal{Z}$ be the center of $\mathcal{N}$. Then $\mathcal{N}$ is represented by the direct sum of $\mathcal{Z}$ and its orthogonal complement $\mathcal{Z}^\perp$.

For each $Z \in \mathcal{Z}$, a skew symmetric linear transformation $j(Z) : \mathcal{Z}^\perp \to \mathcal{Z}^\perp$ is defined by $j(Z)X = (adX)^*Z$ for $X \in \mathcal{Z}^\perp$. Or, equivalently,

$$< j(Z)X, Y >=< [X,Y], Z >$$

for all $X, Y \in \mathcal{Z}^\perp$.

A 2-step nilpotent Lie algebra $\mathcal{N}$ is said to be *an algebra of Heisenberg type* if

$$j(Z)^2 = -|Z|^2 id$$

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for all $Z \in \mathcal{Z}$. And a Lie group N is said to be *a group of Heisenberg type* if its Lie algebra $\mathcal{N}$ is an algebra of Heisenberg type.

The Heisenberg groups are examples of Heisenberg type. That is, let $n \geq 1$ be any integer and $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ a basis of $R^{2n} = \mathcal{V}$. Let $\mathcal{Z}$ be a one dimensional vector space spanned by $\{Z\}$. Define

$$[X_i, Y_i] = -[Y_i, X_i] = Z$$

for any $i = 1, 2, \cdots, n$ with all other brackets are zero. Give on $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ the inner product such that the set of vectors $\{X_i, Y_i, Z | i = 1, 2, \cdots, n\}$ forms an orthonormal basis. Let N be the simply connected 2-step nilpotent group of Heisenberg type which is determined by $\mathcal{N}$ and equipped with a left-invariant metric induced by the inner product in $\mathcal{N}$. The group N is called the $(2n+1)$-dimensional Heisenberg group and denoted by $\mathbb{H}^{2n+1}$.

In this paper, we calculate the volumes of the geodesic balls in the Heisenberg group $\mathbb{H}^5$:

Main Theorem. Let $B_e(R)$ be the geodesic ball with center e (the identity of $\mathbb{H}^5$) and radius R in $\mathbb{H}^5$. Then, the following holds.

$$\begin{aligned} &Vol(B_e(R)) \\ =&\frac{4\pi^2}{6!}\Bigg(p_1(R) + p_4(R)\sin R + p_5(R)\cos R + p_6(R)\int_0^R \frac{\sin t}{t}dt \\ &\qquad + q_4(R)\sin(2R) + q_5(R)\cos(2R) + q_6(R)\int_0^{2R} \frac{\sin t}{t}dt\Bigg) \end{aligned}$$

where

$$\begin{aligned} p_1(R) &= 576R\,, & q_4(R) &= 30 + 54R^2 + 4R^4\,, \\ p_4(R) &= -240 - 108R^2 - 2R^4\,, & q_5(R) &= 132R + 116R^3 + 8R^5\,, \\ p_5(R) &= -528R - 116R^3 - 2R^5\,, & q_6(R) &= 360R^2 + 240R^4 + 16R^6\,, \\ p_6(R) &= -720R^2 - 120R^4 - 2R^6\,. \end{aligned}$$

For a Riemannian manifold M and $p \in M$, the volume growth, $VG_p(M)$ of M at p is defined by

$$VG_p(M) = inf\{x \in R | \lim_{r->\infty} \frac{Vol(B_p(r))}{r^x} = 0\}.$$

If M is a Lie group with a left invariant metric, then we see that $VG_p(M) = VG_q(M)$ for any $p, q \in M$. In this case, it is denoted by $VG(M)$. For example, the volume growth of the Euclidean space $\mathbb{R}^5$ is $VG(\mathbb{R}^5) = 5$ since the volume of ball with radius r is $\frac{8\pi^2 r^5}{15}$ in $\mathbb{R}^5$(see[12]).

Corollary. The volume growth, $VG(\mathbb{H}^5)$ of $\mathbb{H}^5$ is given as follows;

$$VG(\mathbb{H}^5) = 6.$$

2. Preliminaries

Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $<,>$ and N be its unique simply connected 2-step nilpotent Lie group with the left invariant metric induced by $<,>$ on $\mathcal{N}$. The center of $\mathcal{N}$ is denoted by $\mathcal{Z}$. Then $\mathcal{N}$ can be expressed as the direct sum of $\mathcal{Z}$ and its orthogonal complement $\mathcal{Z}^\perp$.

Recall that for $Z \in \mathcal{Z}$, a skew symmetric linear transformation $j(Z) : \mathcal{Z}^\perp \to \mathcal{Z}^\perp$ is defined by $j(Z)X = (\mathrm{ad}X)^* Z$ for $X \in \mathcal{Z}^\perp$. Or, equivalently,

$$\langle j(Z)X, Y\rangle = \langle [X, Y], Z\rangle$$

for $X, Y \in \mathcal{Z}^\perp$. A 2-step nilpotent Lie group N is said to be *a group of Heisenberg type* if

$$j(Z)^2 = -|Z|^2 \,\mathrm{id}$$

for all $Z \in \mathcal{Z}$.

Let $\gamma(t)$ be a curve in N such that $\gamma(0) = e$ (identity element in N) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^\perp$ and $Z_0 \in \mathcal{Z}$. Since $\exp : \mathcal{N} \to N$ is a diffeomorphism ([10]), the curve $\gamma(t)$ can be expressed uniquely by $\gamma(t) = \exp[X(t) + Z(t)]$ with

$$\begin{array}{lll} X(t) \in \mathcal{Z}^\perp, & X'(0) = X_0, & X(0) = 0 \\ Z(t) \in \mathcal{Z}, & Z'(0) = Z_0, & Z(0) = 0. \end{array}$$

A. Kaplan([8],[9]) shows that the curve $\gamma(t)$ is a geodesic in N if and only if

$$\begin{array}{l} X''(t) = j(Z_0)X'(t), \\ Z'(t) + \frac{1}{2}[X'(t), X(t)] \equiv Z_0. \end{array}$$

The following Lemma is useful in the later.

Lemma 2.1 ([2]). *Let N be a simply connected 2-step nilpotent Lie group with a left invariant metric and let $\gamma(t)$ be a geodesic of N with $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^\perp$ and $Z_0 \in \mathcal{Z}$. Then, one has*

$$\gamma'(t) = dl_{\gamma(t)}(X'(t) + Z_0), t \in R$$

where $X'(t) = e^{tj(Z_0)}X_0$ and $l_{\gamma(t)}$ is the left translation by $\gamma(t)$.

Throughout this paper, different tangent spaces will be identified with $\mathcal{N}$ via a left translation. So, in the above lemma, we can consider $\gamma'(t)$ as

$$\gamma'(t) = X'(t) + Z_0 = e^{tj(Z_0)}X_0 + Z_0.$$

Let $\mathbb{H}^{2n+1}$ be the $(2n+1)$-dimensional Heisenberg group with a left invariant metric and $\mathcal{N}$ its Lie algebra. Let $\gamma(t)$ be a unit speed geodesic on $\mathbb{H}^{2n+1}$ with $\gamma(0) = e$(the identity element of $\mathbb{H}^{2n+1}$) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$. Assume that $X_0 \neq 0$ and $Z_0 \neq 0$. Since

$$\{X_0 + Z_0, \frac{|Z_0|}{|X_0|}X_0 - \frac{|X_0|}{|Z_0|}Z_0, \frac{1}{|Z_0||X_0|}j(Z_0)X_0\}$$

is an orthonormal set in $\mathcal{N}$, we can obtain an orthonormal basis

$$\begin{aligned}\mathcal{B} =&\{X_0 + Z_0, \frac{|Z_0|}{|X_0|}X_0 - \frac{|X_0|}{|Z_0|}Z_0, \frac{1}{|Z_0||X_0|}j(Z_0)X_0,\\ &Y_k, \frac{1}{|Z_0|}j(Z_0)Y_k | Y_k \in \mathcal{Z}^{\perp}, k = 1, 2, \cdots, n-1\ \}\end{aligned}$$

by adding

$$\{Y_k, \frac{1}{|Z_0|}j(Z_0)Y_k | Y_k \in \mathcal{Z}^{\perp}, k = 1, 2, \cdots, n-1\}$$

to

$$\{X_0 + Z_0, \frac{|Z_0|}{|X_0|}X_0 - \frac{|X_0|}{|Z_0|}Z_0, \frac{1}{|Z_0||X_0|}j(Z_0)X_0\}.$$

Let

$$\begin{aligned}e_1(t) &= \frac{|Z_0|}{|X_0|}X'(t) - \frac{|X_0|}{|Z_0|}Z_0,\\ e_2(t) &= \frac{1}{|Z_0||X_0|}j(Z_0)X'(t)\end{aligned}$$

and let

$$\begin{aligned}e_{2k-1}(t) &= e^{tj(Z_0)}Y_k,\\ e_{2k}(t) &= \frac{1}{|Z_0|}e^{tj(Z_0)}j(Z_0)Y_k \text{ for each } k = 2, 3, \cdots, n.\end{aligned}$$

Then, $\{\gamma'(t), e_{2k-1}(t), e_{2k}(t) | k = 1, 2, \cdots, n\}$ is an orthonormal frame along $\gamma(t)$ on $\mathbb{H}^{2n+1}$(see[5]). We start the following Proposition.

PROPOSITION 2.2 ([5]). *For each* $k = 1, 2, \cdots, n$, *let* $J_{2k-1}(t)$ *and* $J_{2k}(t)$ *be the Jacobi fields with* $J_{2k-1}(0) = J_{2k}(0) = 0, J'_{2k-1}(0) = e_{2k-1}(0)$ *and* $J'_{2k}(0) = e_{2k}(0)$. *Then, we have that*

(1) for $k = 1$,

$$\begin{bmatrix} J_1(t) \\ J_2(t) \end{bmatrix} = B_1(t) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

where

$$B_1(t) = \frac{1}{|Z_0|^3} \begin{bmatrix} \sin(|Z_0|t) - (1 - |Z_0|^2)|Z_0|t & |Z_0|(\cos(|Z_0|t) - 1) \\ |Z_0|(1 - \cos(|Z_0|t)) & |Z_0|^2 \sin(|Z_0|t) \end{bmatrix},$$

(2) for $k = 2, 3, \cdots, n$

$$\begin{bmatrix} J_{2k-1}(t) \\ J_{2k}(t) \end{bmatrix} = B_k(t) \begin{bmatrix} e_{2k-1}(t) \\ e_{2k}(t) \end{bmatrix}$$

where

$$B_k(t) = \begin{bmatrix} \frac{1}{|Z_0|} \sin(|Z_0|t) & |Z_0|(\cos(|Z_0|t) - 1) \\ \frac{1}{|Z_0|^3}(1 - \cos(|Z_0|t)) & \frac{1}{|Z_0|} \sin(|Z_0|t) \end{bmatrix}.$$

COROLLARY 2.3 ([1],[6]). *Let* $\mathbb{H}^{2n+1}$ *be the* $(2n+1)$*-dimensional Heisenberg group and* $\mathcal{N}$ *its Lie algebra. Let* $\gamma(t)$ *be a unit speed geodesic on* N *with* $\gamma(0) = e$ *(the identity element of* N*) and* $\gamma'(0) = X_0 + Z_0$ *where* $X_0 \in \mathcal{Z}^\perp$ *and* $Z_0 \in \mathcal{Z}$.

(1) If $Z_0 \neq 0$, *then all the conjugate points along* γ *are at* $t \in \frac{2\pi}{|Z_0|}\mathbb{Z}^* \cup \mathbb{A}$ *where*

$$\mathbb{Z}^* = \{\pm 1, \pm 2, \ldots\}$$

and

$$\mathbb{A} = \{t \in \mathbb{R} - \{0\} | (1 - |Z_0|^2)\frac{|Z_0|t}{2} = \tan\frac{|Z_0|t}{2}\}.$$

In particular, $\frac{2\pi}{|Z_0|}$ *is the first conjugate point of* e *along* γ.

(2) If $Z_0 = 0$, *then there are no conjugate points along* γ.

For the conjugate points of another type of Heisenberg groups, Quaternionic Heisenberg groups $\mathbb{H}^{4n+3}$, see [4].

G.Walschap([11]) showed that the first conjugate loci and the cut loci are equal in the case of the groups of Heisenberg type or the 2-step nilpotent groups with a one-dimensional center. So, we consider the geodesic balls $B_e(R)$ with the radius $R \leq 2\pi$. In this paper we calculate the volumes of geodesic balls in $\mathbb{H}^5$.

In ([5]), C. Jang, J. Park and K. Park obtained a formula of the volumes of geodesic balls in the Heisenberg group $\mathbb{H}^3$ as the form of power series.

THEOREM 2.4. ([5] Theorem 3.8) *Let $B_e(R)$ be the geodesic ball with center e and radius R in $\mathbb{H}^3$. Then, the following holds.*

$$Vol(B_e(R)) = 4\pi\left(\frac{R^3}{3} + 2\sum_{n=2}^{\infty}(-1)^n \frac{R^{2n+1}}{(2n+1)!(2n-1)(2n-3)}\right).$$

Recently S. Jeong and K. Park ([7]) calculated the volumes of geodesic balls in the Heisenberg group $\mathbb{H}^3$.

THEOREM 2.5. ([7] Theorem 3.4) *Let $0 \leq R \leq 2\pi$ and $B_e(R)$ be the geodesic ball with center e (the identity of $\mathbb{H}^3$) and radius R in $\mathbb{H}^3$. Then, the following holds.*

$$\begin{aligned} &Vol(B_e(R)) \\ =&\frac{\pi}{6}\left(-16R + (R^2+6)\sin R + (R^3+10R)\cos R + (R^4+12R^2)\int_0^R \frac{\sin t}{t}dt\right). \end{aligned}$$

3. Proof of main theorem

We start to prove Main Theorem. Note that

$$\det(B_1(t)) = \frac{1}{|Z_0|^4}\{2(1-\cos(|Z_0|t)) - (1-|Z_0|^2)|Z_0|t\sin(|Z_0|t)\}$$

and

$$\det(B_k(t)) = \frac{2}{|Z_0|^2}(1-\cos(|Z_0|t))$$

for each $k = 2, 3, \cdots, n$.

LEMMA 3.1 ([5]). *For $t > 0$, the following holds.*

$$\begin{aligned} &\det(B_1(t)B_2(t)) \\ =&\frac{1}{|Z_0|^4}\{2(1-\cos(|Z_0|t)) - (1-|Z_0|^2)|Z_0|t\sin(|Z_0|t)\}\frac{2}{|Z_0|^2}(1-\cos(|Z_0|t)) \geq 0. \end{aligned}$$

LEMMA 3.2 ([7]). *Let n be a natural number and $f : [0,x] \to R$ has continuous $n-th$ derivatives. Assume that for $k = 0, 1, \cdots, n-1$, the $\lim_{t\to 0^+} \frac{f^{(k)}(t)}{t^{n-k}}$ exists and $\frac{f^{(n)}(t)}{t}$ is integrable on $[0,x]$. Then, $\frac{f(t)}{t^{n+1}}$ is integrable on $[0,x]$ and the following holds.*

$$\int_0^x \frac{f(t)}{t^{n+1}} dt = -\sum_{k=0}^{n-1} \frac{1}{n(n-1)\cdots(n-k)} \left[\frac{f^{(k)}(t)}{t^{n-k}}\right]_{0^+}^{x^-} + \frac{1}{n!}\int_0^x \frac{f^{(n)}(t)}{t} dt$$

where

$$\left[\frac{f^{(k)}(t)}{t^{n-k}}\right]_{0^+}^{x^-} = \lim_{t\to x^-} \frac{f^{(k)}(t)}{t^{n-k}} - \lim_{t\to 0^+} \frac{f^{(k)}(t)}{t^{n-k}}.$$

We give a modification of Lemma 3.2 which is useful.

LEMMA 3.3. *Let n be a natural number and $f : [0,1] \to R$ has continuous $n-th$ derivatives. (i) $\forall k = 0, 1, \cdots, n-1$, $f^{(k)}(0) = 0$ and (ii) $\frac{f^{(n)}(t)}{t}$ is integrable on $[0,1]$. Then, $\frac{f(t)}{t^{n+1}}$ is integrable on $[0,1]$ and*

$$\int_0^1 \frac{f(t)}{t^{n+1}} dt = \frac{1}{n!}\Big(-\sum_{k=0}^{n-1}(n-k-1)! f^{(k)}(1) + f^{(n)}(0)\sum_{k=1}^{n}\frac{1}{k} + \int_0^1 \frac{f^{(n)}(t)}{t} dt\Big).$$

Proof. We use the LHopitals theorem.

$$\begin{aligned}\lim_{t\to 0^+} \frac{f(t)}{t^n} &= \lim_{t\to 0^+} \frac{f^{(1)}(t)}{nt^{n-1}} = \lim_{t\to 0^+} \frac{f^{(2)}(t)}{n(n-1)t^{n-2}} = \cdots \\ &= \lim_{t\to 0^+} \frac{f^n(t)}{n(n-1)\cdots(n-n+1)t^{n-n}}.\end{aligned}$$

We see that $\lim_{t\to 0^+} \frac{f(t)}{t^n} = \frac{f^{(n)}(0)}{n!}$. Let $\alpha = \frac{f^{(n)}(0)}{n!}$, then

$$\lim_{t\to 0^+} \frac{f^{(k)}(t)}{t^{n-k}} = n(n-1)\cdots(n-k+1)\alpha.$$

By Lemma 3.2, we have that

$$
\begin{aligned}
&\int_0^1 \frac{f(t)}{t^{n+1}}dt \\
=&-\sum_{k=0}^{n-1}\frac{1}{n(n-1)\cdots(n-k)}\left(\lim_{t\to 1^-}\frac{f^{(k)}(t)}{t^{n-k}}-\lim_{t\to 0^+}\frac{f^{(k)}(t)}{t^{n-k}}\right)\\
&+\frac{1}{n!}\int_0^1\frac{f^{(n)}(t)}{t}dt\\
=&-\sum_{k=0}^{n-1}\frac{1}{n(n-1)\cdots(n-k)}\left(f^{(k)}(1)-n(n-1)\cdots(n-k+1)\cdot\alpha\right)\\
&+\frac{1}{n!}\int_0^1\frac{f^{(n)}(t)}{t}dt\\
=&-\sum_{k=0}^{n-1}\frac{f^{(k)}(1)}{n(n-1)\cdots(n-k)}+\sum_{k=0}^{n-1}\frac{\alpha}{n-k}+\frac{1}{n!}\int_0^1\frac{f^{(n)}(t)}{t}dt\\
=&-\sum_{k=0}^{n-1}\frac{f^{(k)}(1)}{n(n-1)\cdots(n-k)}+\frac{f^{(n)}(0)}{n!}\sum_{k=1}^{n}\frac{1}{k}+\frac{1}{n!}\int_0^1\frac{f^{(n)}(t)}{t}dt\\
=&-\sum_{k=0}^{n-1}\frac{(n-k-1)!f^{(k)}(1)}{n(n-1)\cdots(n-k)(n-k-1)!}+\frac{f^{(n)}(0)}{n!}\sum_{k=1}^{n}\frac{1}{k}+\frac{1}{n!}\int_0^1\frac{f^{(n)}(t)}{t}dt\\
=&\frac{1}{n!}\left(-\sum_{k=0}^{n-1}(n-k-1)!f^{(k)}(1)+f^{(n)}(0)\sum_{k=1}^{n}\frac{1}{k}+\int_0^1\frac{f^{(n)}(t)}{t}dt\right).
\end{aligned}
$$

This completes the proof. □

We introduce the volume formula of geodesic balls in Riemannian manifolds, which is well-known. For example, see ([3]). Let M be a Riemannian manifold with a metric g and $p \in M$. Take an orthonormal basis $\{u_1, u_2, \cdots, u_n\}$ of T_pM and let $(x_1, x_2, \cdots, x_n)$ be the coordinates determined by $\{u_1, u_2, \cdots, u_n\}$. This local coordinate system is called the normal coordinate system at p. It is easy to show that $\frac{\partial}{\partial x_i}_m = (d\exp_p)_{\sum_{i=1}^n x_iu_i}(u_i)$ where $m = \exp_p(\sum_{i=1}^n x_iu_i)$. Then, the volume form v_g on U_p is given by

$$v_g = \sqrt{\det\left(g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\right)}dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

where g_{ij} is the metric coefficients of g in U_p. Therefore, the volume of the geodesic ball $B_p(r)$ is given by

$$Vol(B_p(r)) = \int_{\exp_p^{-1}(B_p(r))} \exp_p^* v_g.$$

Let $\gamma(t)$ be the unit speed geodesic in M with $\gamma(0) = p$, $\gamma'(0) = u_1$ and let $J_i(t)$ be the Jacobi field with $J_i(0) = 0$ and $J_i'(0) = u_i$ for each $i = 2, 3, \cdots, n$. Then we know that

$$(d\exp_p)_{tu_1} u_1 = \gamma'(t)$$

and

$$(d\exp_p)_{tu_1} u_i = \frac{1}{t} J_i(t)$$

for each $i = 2, 3, \cdots, n$. So, we see that

$$\sqrt{\det\left(g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\right)} = t^{-(n-1)}\sqrt{\det(g(J_i(t), J_j(t)))}.$$

Hence, we have that

$$\exp_p^* v_g = t^{-(n-1)}\sqrt{\det(g(J_i(t), J_j(t)))}dx_1 dx_2 \cdots dx_n = \sqrt{\det(g(J_i(t), J_j(t)))}dtdu$$

where du denotes the canonical measure of the unit sphere S^{n-1}. Therefore, by Fubini's Theorem we get that

$$Vol(B_p(r)) = \int_{S^{n-1}} \int_0^r \sqrt{\det(g(J_i(t), J_j(t)))}dtdu.$$

Using Proposition 2.2 for n=2 , we obtain that

$$\begin{aligned}
&\det\left(< J_i(t), J_j(t) >\right) \\
&= \det\left(J_i(t) \cdot J_j(t)\right) \\
&= \det\left(\begin{bmatrix} J_1(t) \\ J_2(t) \\ J_3(t) \\ J_4(t) \end{bmatrix} \begin{bmatrix} J_1(t) & J_2(t) & J_3(t) & J_4(t) \end{bmatrix}\right) \\
&= \det\left(B_1(t)\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \cdot^t \left(B_1(t)\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}\right)\right) \det\left(B_2(t)\begin{bmatrix} e_3(t) \\ e_4(t) \end{bmatrix} \cdot^t \left(B_2(t)\begin{bmatrix} e_3(t) \\ e_4(t) \end{bmatrix}\right)\right) \\
&= \det\left(B_1(t) \cdot^t (B_1(t))\right) \det\left(B_2(t) \cdot^t (B_2(t))\right) \\
&= \left(\frac{1}{|Z_0|^4}\{2(1-\cos(|Z_0|t)) - (1-|Z_0|^2)|Z_0|t\sin(|Z_0|t)\}\{\frac{2}{|Z_0|^2}(1-\cos(|Z_0|t))\}\right)^2.
\end{aligned}$$

By Lemma 3.1, we have that

$$\sqrt{\det\left(< J_i(t), J_j(t) >\right)}$$

$$=\frac{1}{|Z_0|^4}\{2(1-\cos(|Z_0|t)) - (1-|Z_0|^2)|Z_0|t\sin(|Z_0|t)\}\{\frac{2}{|Z_0|^2}(1-\cos(|Z_0|t))\}.$$

Let $u = (x_1, x_2, x_3, x_4, x_5) \in S^4$ and $|Z_0| = x_5$, then

$$f(x_5, t) = \frac{1}{(x_5)^4}\{2(1-\cos(x_5 t)) - (1-(x_5)^2)x_5 t\sin(x_5 t)\}\{\frac{2}{(x_5)^2}(1-\cos(x_5 t))\}.$$

Therefore, we see that volume of geodesic ball with center e (the identity of $\mathbb{H}^5$) and radius R in $\mathbb{H}^5$ is given as follows;

$$Vol(B_e(R)) = \int_{S^4}\int_0^R f(x_5, t)dtdu.$$

Since the area element du on the sphere S^4 is given by

$$du = \frac{1}{\sqrt{1-(x_1^2+x_2^2+x_3^2+x_4^2)}}dx_1dx_2dx_3dx_4,$$

we have that

$$Vol(B_e(R)) = 2\int_D\int_0^R f(\sqrt{1-(x_1^2+x_2^2+x_3^2+x_4^2)}, t)\frac{dtdx_1dx_2dx_3dx_4}{\sqrt{1-(x_1^2+x_2^2+x_3^2+x_4^2)}}$$

where

$$D = \{(x_1, x_2, x_3, x_4) | x_1^2+x_2^2+x_3^2+x_4^2 \leq 1\}.$$

Changing the coordinates on D to spherical coordinates, we have that

$$Vol(B_e(R))$$

$$=2\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\int_0^1\int_0^R f(\sqrt{1-r^2}, t)\frac{1}{\sqrt{1-r^2}}\cdot r^3\sin^2\theta_1\sin\theta_2 dtdrd\theta_1 d\theta_2 d\theta_3$$

$$=4\pi\prod_{k=1}^{2}\int_0^{\pi}\sin^k\theta d\theta\int_0^1\int_0^R\frac{r^3}{\sqrt{1-r^2}}f(\sqrt{1-r^2}, t)dtdr$$

$$=4\pi^2\int_0^1\int_0^R\frac{r^3}{\sqrt{1-r^2}}f(\sqrt{1-r^2}, t)dtdr.$$

Replacing $x = \sqrt{1-r^2}$, we see that

$$Vol(B_e(R)) = 4\pi^2\int_0^1(1-x^2)\int_0^R f(x, t)dtdx$$

where $f(x,t) = \frac{1}{x^4}\{2(1-\cos(xt)) - (1-x^2)xt\sin(xt)\}\{\frac{2}{x^2}(1-\cos(xt))\}$.
Since

$$\int_0^R f(x,t)dt = \frac{1}{x^6}\Big(-R(1-\cos(Rx))^2 + 5\int_0^R (1-\cos(xt))^2 dt\Big) + \frac{1}{x^4}\Big(R(1-\cos(Rx))^2 - \int_0^R (1-\cos(xt))^2 dt\Big).$$

We have that

$$\begin{aligned} &Vol(B_e(R)) \\ &= 4\pi^2 \int_0^1 (1-x^2)\int_0^R f(x,t)dtdx \\ &= 4\pi^2\Big[\int_0^1 \frac{1-x^2}{x^6}\Big(-R(1-\cos(Rx))^2 + 5\int_0^R (1-\cos(xt))^2 dt\Big)dx \\ &\quad + \int_0^1 \frac{1-x^2}{x^4}\Big(R(1-\cos(Rx))^2 - \int_0^R (1-\cos(xt))^2 dt\Big)dx\Big]. \end{aligned}$$

By the following Lemma 3.4, we see that

$$\begin{aligned} &Vol(B_e(R)) \\ &= \frac{4\pi^2}{6!}\Big(576R + (-240 - 108R^2 - 2R^4)\sin R + (-528R - 116R^3 - 2R^5)\cos R \\ &\quad + (30 + 54R^2 + 4R^4)\sin(2R) + (132R + 116R^3 + 8R^5)\cos(2R) \\ &\quad + (-720R^2 - 240R^4 + 2R^6)\int_0^R \frac{\sin t}{t}dt \\ &\quad + (360R^2 + 480R^4 - 16R^6)\int_0^{2R} \frac{\sin t}{t}dt\Big). \end{aligned}$$

We can rewrite the above equation as follows

$$\begin{aligned} &Vol(B_e(R)) \\ &= \frac{4\pi^2}{6!}\Big(p_1(R) + p_4(R)\sin R + p_5(R)\cos R + p_6(R)\int_0^R \frac{\sin t}{t}dt \\ &\quad + q_4(R)\sin(2R) + q_5(R)\cos(2R) + q_6(R)\int_0^{2R}\frac{\sin t}{t}dt\Big) \end{aligned}$$

where

$$\begin{aligned} p_1(R) &= 576R\,, & q_4(R) &= 30+54R^2+4R^4\,,\\ p_4(R) &= -240-108R^2-2R^4\,, & q_5(R) &= 132R+116R^3+8R^5\,,\\ p_5(R) &= -528R-116R^3-2R^5\,, & q_6(R) &= 360R^2+240R^4+16R^6\,,\\ p_6(R) &= -720R^2-120R^4-2R^6\,. \end{aligned}$$

Thus we finish the proof.

LEMMA 3.4. *For $R>0$, the followings hold.*

(1) $$\int_0^1 \frac{1-x^2}{x^6}\left(-R(1-\cos(Rx))^2+5\int_0^R(1-\cos(xt))^2dt\right)dx$$
$$\begin{aligned}=\frac{1}{6!}\Bigg(&576R+(-600+72R^2-2R^4)\sin R+(-168R+64R^3-2R^5)\cos R\\ &+(75-36R^2+4R^4)\sin(2R)+(42R-64R^3+8R^5)\cos(2R)\\ &+\int_0^1\frac{(60R^4-2R^6)\sin(Rx)+(-120R^4+16R^6)\sin(2Rx)}{x}dx\Bigg).\end{aligned}$$

(2) $$\int_0^1 \frac{1-x^2}{x^4}\left(R(1-\cos(Rx))^2-\int_0^R(1-\cos(xt))^2dt\right)dx$$
$$\begin{aligned}=\frac{1}{4!}\Bigg(&(12-6R^2)\sin R+(-12R-6R^3)\cos R\\ &+(-\frac{3}{2}+3R^2)\sin(2R)+(3R+6R^3)\cos(2R)\\ &+\int_0^1\frac{(-24R^2-6R^4)\sin(Rx)+(12R^2+12R^4)\sin(2Rx)}{x}dx\Bigg).\end{aligned}$$

Proof. Let $h(x)=\int_0^{Rx}(1-\cos t)^2dt$. Then, we have that

$$\begin{aligned} h'(x) &= R(1-\cos(Rx))^2\,, & h''(x) &= 2R^2\sin(Rx)-R^2\sin(2Rx)\,,\\ h^{(3)}(x) &= 2R^3\cos(Rx)-2R^3\cos(2Rx)\,, & h^{(4)}(x) &= -2R^4\sin(Rx)+4R^4\sin(2Rx)\,,\\ h^{(5)}(x) &= -2R^5\cos(Rx)+8R^5\cos(2Rx)\,, & h^{(6)}(x) &= 2R^6\sin(Rx)-16R^6\sin(2Rx)\,. \end{aligned}$$

So, we can rewrite that

$$\begin{aligned}&\int_0^1 \frac{1-x^2}{x^6}\left(-R(1-\cos(Rx))^2+5\int_0^R(1-\cos(xt))^2dt\right)dx\\ &=\int_0^1\frac{1-x^2}{x^7}(-xh'(x)+5h(x))dx\end{aligned}$$

and

$$\int_0^1 \frac{1-x^2}{x^4}\left(R(1-\cos(Rx))^2 - \int_0^R (1-\cos(xt))^2 dt\right)dx$$
$$= \int_0^1 \frac{1-x^2}{x^5}(xh'(x) - h(x))dx.$$

To compute the above integral, the following derivatives are useful . Let $m(x) = (1-x^2)(-xh'(x) + 5h(x))$. Then, derivatives of $m(x)$ give that

$$\begin{aligned}
m'(x) =& -2x(-xh'(x)+5h(x)) + (1-x^2)(-xh''(x)+4h'(x))\\
m''(x) =& -2(-xh'(x)+5h(x)) - 4x(-xh''(x)+4h'(x))\\
&+ (1-x^2)(-xh^{(3)}(x)+3h''(x))\\
m^{(3)}(x) =& -6(-xh''(x)+4h'(x)) - 6x(-xh^{(3)}(x)+3h''(x))\\
&+ (1-x^2)(-xh^{(4)}(x)+2h^{(3)}(x))\\
m^{(4)}(x) =& -12(-xh^{(3)}(x)+3h''(x)) - 8x(-xh^{(4)}(x)+2h^{(3)}(x))\\
&+ (1-x^2)(-xh^{(5)}(x)+h^{(4)}(x))\\
m^{(5)}(x) =& -20(-xh^{(4)}(x)+2h^{(3)}(x)) - 10x(-xh^{(5)}(x)+h^{(4)}(x))\\
&+ (1-x^2)(-xh^{(6)}(x))\\
m^{(6)}(x) =& -30(-xh^{(5)}(x)+h^{(4)}(x)) - 12x(-xh^{(6)}(x))\\
&+ (1-x^2)(-xh^{(7)}(x)-h^{(6)}(x)).
\end{aligned}$$

Let $q(x) = (1-x^2)(xh'(x) - h(x))$. Then, derivatives of $q(x)$ give that

$$\begin{aligned}
q'(x) =& -2x(xh'(x)-h(x)) + (1-x^2)(xh''(x))\\
q''(x) =& -2(xh'(x)-h(x)) - 4x(xh''(x)) + (1-x^2)(h''(x)+xh^{(3)}(x))\\
q^{(3)}(x) =& -6(xh''(x)) - 6x(h''(x)+xh^{(3)}(x)) + (1-x^2)(2h^{(3)}(x)+xh^{(4)}(x))\\
q^{(4)}(x) =& -12(h''(x)+xh^{(3)}(x)) - 8x(2h^{(3)}(x)+xh^{(4)}(x))\\
&+ (1-x^2)(3h^{(4)}(x)+xh^{(5)}(x)).
\end{aligned}$$

By the above substitutions, we can obtain the following results. We can rewrite that

$$\int_0^1 \frac{1-x^2}{x^7}(-xh'(x)+5h(x))dx = \int_0^1 \frac{m(x)}{x^7}dx$$

and

$$\int_0^1 \frac{1-x^2}{x^5}(xh'(x)-h(x))dx = \int_0^1 \frac{q(x)}{x^5}dx.$$

Using Lemma 3.3, we have that

$$\begin{aligned}
&\int_0^1 \frac{m(x)}{x^7}dx \\
=&\frac{1}{6!}\left(-\sum_{k=0}^{5}(5-k)!m^{(k)}(1)+m^{(6)}(0)\sum_{k=1}^{6}\frac{1}{k}+\int_0^1 \frac{m^{(6)}(t)}{t}dt\right) \\
=&\frac{1}{6!}\Bigg(300h(1)+84h^{(1)}(1)+36h^{(2)}(1)+32h^{(3)}(1)+h^{(4)}(1)+h^{(5)}(1) \\
&\quad+\int_0^1 \frac{-30h^{(4)}(x)-h^{(6)}(x)}{x}dx\Bigg) \\
=&\frac{1}{6!}\Bigg(576R+(-600+72R^2-2R^4)\sin R+(-168R+64R^3-2R^5)\cos R \\
&\quad+(75-36R^2+4R^4)\sin(2R)+(42R-64R^3+8R^5)\cos(2R) \\
&\quad+\int_0^1 \frac{(60R^4-2R^6)\sin(Rx)+(-120R^4+16R^6)\sin(2Rx)}{x}dx\Bigg).
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \frac{q(x)}{x^5}dx \\
=&\frac{1}{4!}\left(-\sum_{k=0}^{3}(3-k)!q^{(k)}(1)+q^{(4)}(0)\sum_{k=1}^{4}\frac{1}{k}+\int_0^1 \frac{q^{(4)}(t)}{t}dt\right) \\
=&\frac{1}{4!}\left(-6h(1)+6h^{(1)}(1)-3h^{(2)}(1)-3h^{(3)}(1)+\int_0^1 \frac{-12h^{(2)}(x)+3h^{(4)}(x)}{x}dx\right) \\
=&\frac{1}{4!}\Bigg((12-6R^2)\sin R+(-12R-6R^3)\cos R \\
&\quad+(-\frac{3}{2}+3R^2)\sin(2R)+(3R+6R^3)\cos(2R) \\
&\quad+\int_0^1 \frac{(-24R^2-6R^4)\sin(Rx)+(12R^2+12R^4)\sin(2Rx)}{x}dx\Bigg).
\end{aligned}$$

This completes the proof. □

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